The CASTLE

Stuyvesant Math Team

December 18, 2024

The CASTLE is a 3-hour contest meant to resemble the AIME in difficulty and format. It stands for Corwin Andy Steven Tanvir Lucas Examination! All answers are integers from 000 to 999, inclusive. Good luck and enjoy!

Problem 1. A unit circle is in the coordinate plane with its center at the origin. There are 20 equally spaced points on the perimeter of the circle such that one of the points is at $(1, 0)$. How many lines passing through two of the points intersect the y-axis at a positive y-value (excluding the y-axis itself)?

Problem 2. Find the unique positive integer $a \geq 100$ such that the coefficients of x^{100} and x^{101} are equal in the expansion of $(ax + 2025)^a$.

Problem 3. Regular square pyramid $\mathcal{P}$ has apex A and square base $\mathcal{S}_1$. Parallel to $\mathcal{S}_1$ is another square base $\mathcal{S}_2$ with its edges on the pyramid. If the shape consisting of the section of $\mathcal{P}$ bounded by $\mathcal{S}_1$ and $\mathcal{S}_2$ (including both $\mathcal{S}_1$ and $\mathcal{S}_2$) has surface area 360, the pyramid with apex A and base $\mathcal{S}_2$ has surface area 96, and $\mathcal{P}$ has surface area 384, find the volume of $\mathcal{P}$.

Problem 4. Let a *tower* be a nonempty collection of blocks stacked on top of each other. How many ways can you arrange 4 red blocks, 5 blue blocks, and 11 green blocks into two towers such that no block is stacked on top of another block of the same color? Blocks of the same color and towers are indistinguishable.

Problem 5. Find the number of ordered pairs (a, b) of positive integers with $ab = 20^{24}$ such that

$$\log_a(\gcd(a, b)) = \log_{\mathrm{lcm}(a,b)}(b).$$

Problem 6. Find the number of positive integers n less than 1000 such that $10 \mid \lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{5} \rfloor$. Note that for a real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x. For example, $\lfloor 3 \rfloor = \lfloor \pi \rfloor = 3$.

Problem 7. Let $\nu_3(n)$ be the exponent of the largest power of 3 that divides n (for instance, $\nu_3(3) = 1$, $\nu_3(9) = 2$, $\nu_3(33) = 1$). Moreover, let $N = 1 + 3 + 3^2 + \cdots + 3^{99}$. Find the remainder when

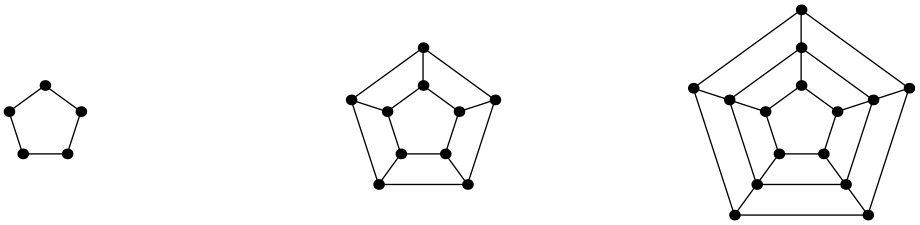
$$\sum_{k=1}^{N} 2^{\nu_3(k)}$$

is divided by 1000.

Problem 8. In triangle ABC, let point D lie on $\overline{BC}$ and ω_1 and ω_2 be the incircles of triangles ABD and ACD, respectively. Let $\overline{AB}$ meet ω_1 at P, $\overline{AC}$ meet ω_2 at Q, and $\overline{AD}$ meet both ω_1 and ω_2 at X. Line PX intersects ω_2 at $R \neq X$ and line QX intersects ω_1 at $S \neq X$. Given that $AX = 12$ and the radii of ω_1 and ω_2 are 5 and 8, respectively, find the area of $PQRS$.

Problem 9. Suppose there are 8 distinct castles positioned on the vertices of a regular octagon. Let a *bridge* be a straight line segment connecting two castles and a *kingdom* be a combination of all 8 castles and a nonzero number of bridges, no two of which intersect (if two bridges share a castle, they are also said to intersect). How many possible kingdoms are there?

Problem 10. Let an *n-pentagon* be the set of vertices and segments in n distinct regular pentagons centered at the same point whose sides are parallel. An *n-web* is an n-pentagon along with segments that connect each vertex to its closest neighbors in each pentagon adjacent to the one containing that vertex. A 1-web, a 2-web, and a 3-web are shown below.



Let N be the number of ways to color each of the vertices of a 10-web with 1 of 3 colors such that no two vertices of the same color are connected by a segment (rotations and reflections of the same coloring are distinct). Find the remainder when N is divided by 1000.

Problem 11. Let $\triangle ABC$ be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$ with circumcircle ω. The line through B parallel to AC meets ω at X, while the line through C parallel to AB meets ω at Y. If P and Q are the reflections of X over AB and Y over AC, respectively, the length PQ can be represented as $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

Problem 12. For a 5×5 grid of lattice points, let $\mathcal{S}$ denote the set of paths from the lower left corner to the upper right corner that do not visit any lattice point twice and consist solely of 1 unit moves up, down, or to the right. The expected value of the length of a path in $\mathcal{S}$ can be written as $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

Problem 13. Let $ABCDEF$ be a convex hexagon satisfying $AB = CD = EF$ inscribed in circle ω. Lines AD and BE meet at P, lines BE and CF meet at Q, and lines CF and AD meet at R. Suppose the circumcircles of triangles PED, QCB, RAF have radii $\frac{2}{9}$, $\frac{1}{3}$, $\frac{2}{3}$. The triangle bounded by lines AB, CD, EF has inradius 1 and area $\frac{p}{q}$ for relatively prime positive integers p and q. Find $p + q$.

Problem 14. For a positive integer n, let $\tau(n)$ be the number of positive integer divisors of n and let $\varphi(n)$ be the number of positive integers less than or equal to n which are relatively prime to n. Find the remainder when the sum of all positive integer solutions to the equation

$$5 \cdot \tau(n) = \varphi(n)$$

is divided by 1000.

Problem 15. The value of the sum

$$\sum_{k=1}^{97} \frac{1}{\left(1 + 2\cos\frac{2\pi k}{97}\right)^2}$$

can be expressed as $\frac{p}{q}$ for relatively prime positive integers p and q. Find the remainder when $p + q$ is divided by 1000.